Optimal Trajectories and Linear Control of Nonlinear Systems

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The previously reported successive-approximations scheme, based on the adjoint system, for the integration of the nonlinear variational two-point boundary-value problem is extended here to problems in which the initial and terminal boundary conditions are general functions of the problem variables, thus permitting the treatment of a general Mayer problem. Inequality constraints, which may, but need not, explicitly depend on the control, are included. The linear relationship (a Green's theorem) used in the successive approximations procedure also yields the linear feedback control law in the neighborhood of an optimal nominal trajectory (neighboring extremal control). The adjoint system is equivalent to previous analyses based on the second variation of the calculus of variations, and the feedback control scheme derived here is similar. The present method of constructing the feedback gains for certain neighboring extremal control problems involves less computation than previous methods. Numerical examples of both the optimization and control schemes are presented for planar rocket maneuvers outside the earth's atmosphere.

Introduction

Numerical schemes of the steepest-descent type are now well known and widely used in obtaining approximate optimal open-loop control programs for nonlinear systems. These schemes involve successive approximations to the control program. They generally converge very adequately on the terminal constraints but often yield control programs which poorly approximate the Eulerian control (the solution of the variational boundary-value problem). In addition, steepest-descent schemes exhibit poor convergence properties as the optimal trajectory is approached, although ingenious modifications have been devised to improve the convergence.[1, 2]

On the other hand, efforts to solve the variational boundary-value problem have met with difficulties because of the extreme sensitivity of the Lagrange multipliers whose initial values have to be determined in successive approximations to the problem. Methods based on numerical differentiation (i.e., repeated integration of the boundary-value problem to determine the effect of changes of initial values of the Lagrange multipliers on terminal boundary conditions) present fundamental difficulties, since differentiation is one of the most inaccurate numerical procedures.

Recently, Jurovics and McIntyre[3] and this author[4] proposed a rapidly convergent successive approximations scheme, based on the adjoint system, for the integration of two-point boundary-value problems. Here the sensitivities are computed by integration. The solutions to the adjoint equations (adjoint to the variational boundary-value problem that includes the equations of state and the Euler-Lagrange equations) are Green's functions that linearly relate perturbations in initial and terminal boundary conditions.† The work previously mentioned is an extension of the work of Goodman and Lance[5] to boundary-value problems with one unknown boundary, which is the form of the variational boundary-value problem. This work is extended here to problems in which the initial and terminal boundary conditions are general functions of the problem variables, thus permitting the treatment of a general Mayer problem. Inequality constraints (which may, but need not, involve the control) are reformulated[6] as terminal constraints in the optimization scheme.

Open-loop control programs are clearly inadequate for actual implementation, since deviations from the optimal trajectory will invariably occur because of external and internal disturbances. A feedback control scheme is required. If an optimal nominal trajectory is predetermined, it is clearly desirable to follow a trajectory that is optimal in the same sense as the nominal when disturbances force the system from the nominal path. Recently, Kelley[7] and Breakwell, Speyer, and Bryson[8] proposed a feedback control scheme that preserves optimality and provides terminal control for small deviations from the optimal nominal path. This scheme involves linear approximation to neighboring extremals (neighboring extremal control) and involves in its development the second variation of the calculus of variations.

The open-loop optimization scheme presented here leads easily to the derivation of the matrix of linear, time-varying feedback gains for neighboring extremal control. The derivation is simple and intuitive. When the open-loop optimization problem is solved, all the components required to construct the feedback gains are available. In comparison with the scheme proposed in Ref. 8, less computation is required in constructing the feedback gains for neighboring extremal control to a neighboring end point, which is the most general form of the control problem treated.

The analysis presented here is formal. The existence and continuity of appropriate derivatives are assumed. In particular, it is assumed that the optimal nominal trajectory is nonsingular, which excludes systems linear in the control.

Numerical examples of both the optimization and control schemes are presented for planar rocket maneuvers outside the earth's atmosphere. Typical convergence properties of the optimization scheme for this type of problem are shown. A simple example demonstrates the inadequacy of steepest descent in generating a Eulerian open-loop control.

I. A General Two-Point Boundary-Value Problem

We formulate the problem in terms of the nonlinear, first-order, ordinary vector differential equation

$$\dot{z} = F(t, z) \tag{1}$$

where z is a $2n$ vector‡ and t the scalar independent variable,

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† References 8 and 11 obtain these sensitivities by integrating the perturbation equations of the boundary value problem.

‡ Vectors are column vectors unless otherwise specified.

which will be referred to as time. The initial boundary t_0 will be assumed known, whereas the terminal boundary t_1 will be an unknown. The $2n + 1$ boundary conditions are

$$g(t_0, z(t_0)) = 0$$
$$h(t_1, z(t_1)) = 0 \tag{2}$$

where g is a p vector and h is a $2n - p + 1$ vector.

Initially, $z(t_0)$ is chosen so as to satisfy the initial boundary conditions $g = 0$, and Eq. (1) is integrated forward ($t_0 \to \bar{t}_1$) to some terminal time $\bar{t}_1$. We thus obtain a solution $\bar{z}(t)$ that, in general, will not satisfy the terminal boundary conditions $h = 0$. We therefore consider linear perturbations from the reference solution $\bar{z}(t)$.

The linear perturbation equations are

$$\delta \dot{z} = [\partial F/\partial z]\delta z \tag{3}$$

where

$$\delta z(t) = z(t) - \bar{z}(t); \left[\frac{\partial F}{\partial z}\right] = \begin{bmatrix} \frac{\partial F_1}{\partial z_1} & \cdots & \frac{\partial F_1}{\partial z_{2n}} \\ \vdots & & \vdots \\ \frac{\partial F_{2n}}{\partial z_1} & \cdots & \frac{\partial F_{2n}}{\partial z_{2n}} \end{bmatrix}_{\bar{z}}$$

and the matrix of partial derivatives is to be evaluated along the reference solution. The equations adjoint to Eq. (3) are

$$\dot{\lambda} = -[\partial F/\partial z]^T \lambda \tag{4§}$$

Equations (3) and (4) possess the property

$$d/dt\ (\lambda^T \delta z) = 0 \tag{5}$$

Integrating Eq. (5) from t_0 to $\bar{t}_1$, we obtain

$$\lambda^T(\bar{t}_1)\delta z(\bar{t}_1) - \lambda^T(t_0)\delta z(t_0) = 0 \tag{6}$$

We wish to vary the terminal boundary $\bar{t}_1$, since in general $\bar{t}_1 \neq t_1$. To the first order, the total perturbations in the dependent variables at $\bar{t}_1$ are given by

$$dz(\bar{t}_1) = \delta z(\bar{t}_1) + \dot{z}(\bar{t}_1)d\bar{t}_1 \tag{7}$$

Equation (6) therefore becomes

$$\lambda^T(\bar{t}_1)[dz(\bar{t}_1) - \dot{z}(\bar{t}_1)d\bar{t}_1] - \lambda^T(t_0)\delta z(t_0) = 0 \tag{8}$$

As yet, no boundary conditions have been specified on the adjoint equation (4). Integrate Eqs. (4) $2n - p + 1$ times backward from $\bar{t}_1$ to t_0 with the initial conditions

$$_1\lambda(\bar{t}_1) = \begin{bmatrix} \frac{\partial h_1}{\partial z_1} \\ \vdots \\ \frac{\partial h_1}{\partial z_{2n}} \end{bmatrix}_{\bar{t}_1} \qquad _2\lambda(\bar{t}_1) = \begin{bmatrix} \frac{\partial h_2}{\partial z_1} \\ \vdots \\ \frac{\partial h_2}{\partial z_{2n}} \end{bmatrix}_{\bar{t}_1} \quad \cdots \quad _{2n-p+1}\lambda(\bar{t}_1) = \begin{bmatrix} \frac{\partial h_{2n-p+1}}{\partial z_1} \\ \vdots \\ \frac{\partial h_{2n-p+1}}{\partial z_{2n}} \end{bmatrix}_{\bar{t}_1} \tag{9}$$

Clearly, Eq. (8) holds for each integration. Consider the first integration. From Eq. (8) we have

$$[\partial h_1/\partial z]^T[dz(\bar{t}_1) - \dot{z}(\bar{t}_1)d\bar{t}_1] - {_1\lambda^T}(t_0)\delta z(t_0) = 0$$

§ Superscript T denotes the transpose.

But

$$\left[\frac{\partial h_1}{\partial z}\right]^T dz(\bar{t}_1) - \left[\frac{\partial h_1}{\partial z}\right]^T \dot{z}(\bar{t}_1)d\bar{t}_1 = dh_1 - \frac{\partial h_1}{\partial t_1}d\bar{t}_1 - \left[\frac{\partial h_1}{\partial z}\right]^T \dot{z}(\bar{t}_1)d\bar{t}_1 = dh_1 - \dot{h}_1(\bar{t}_1)d\bar{t}_1$$

Therefore, from the $2n - p + 1$ integrations we obtain

$$\begin{aligned} \dot{h}_1(\bar{t}_1)d\bar{t}_1 + {_1\lambda^T}(t_0)\delta z(t_0) &= dh_1 \\ &\vdots \\ \dot{h}_{2n-p+1}(\bar{t}_1)d\bar{t}_1 + {_{2n-p+1}\lambda^T}(t_0)\delta z(t_0) &= dh_{2n-p+1} \end{aligned} \tag{10}$$

Each of Eqs. (10) is a Green's theorem in the plane. The components of $_i\lambda(t_0)$ are Green's functions, which relate perturbations in initial values of the dependent variables to the change $dh_i(\bar{t}_1)$. Since Eqs. (10) are based on a linearization about a reference solution, we expect accurate results for sufficiently small perturbations. If the boundary-value problem posed in Eq. (1) were linear, Eqs. (10) would hold exactly for all perturbations $\delta z(t_0)$.

Not all the perturbations $\delta z_i(t_0)$ are independent. They are related through the initial boundary conditions $g = 0$, since it is required that

$$dg(t_0) = [\partial g/\partial z]\delta z(t_0) = 0 \tag{11}$$

where

$$\left[\frac{\partial g}{\partial z}\right] = \begin{bmatrix} \frac{\partial g_1}{\partial z_1} & \cdots & \frac{\partial g_1}{\partial z_{2n}} \\ \vdots & & \vdots \\ \frac{\partial g_p}{\partial z_1} & \cdots & \frac{\partial g_p}{\partial z_{2n}} \end{bmatrix}_{t_0}$$

Equations (11) may be solved for p of $\delta z_i(t_0)$ in terms of the $2n-p$ remaining $\delta z_i(t_0)$.¶ p of the $\delta z_i(t_0)$ is therefore eliminated from Eqs. (10), and these then represent $2n - p + 1$ equations in the $2n - p + 1$ unknowns: $d\bar{t}_1$ and $2n - p$ initial perturbations $\delta z_i(t_0)$, assuming the dh_i to be specified.

We therefore have a systematic way of improving the initial choice of $z(t_0)$ and $\bar{t}_1$ so as to satisfy the desired terminal boundary conditions $h = 0$. Having obtained the reference solution $\bar{z}(t)$, we integrate the adjoint equations (4) $2n - p + 1$ times backward in time with "initial" conditions (9). Next we solve Eqs. (11) for p of the $\delta z_i(t_0)$ and eliminate these from Eqs. (10). We then specify the desired changes in terminal boundary conditions

$$dh_i(\bar{t}_1) = -c_i h_i(\bar{t}_1) \text{ no sum on } i \qquad 0 < c_i \leq 1$$

and solve Eqs. (10) for $d\bar{t}_1$ and the $2n - p$ remaining $\delta z_i(t_0)$. Next obtain an improved reference solution with

$$\bar{t}_1^{\text{new}} = \bar{t}_1^{\text{old}} + d\bar{t}_1$$
$$z(t_0)^{\text{new}} = z(t_0)^{\text{old}} + \delta z(t_0)$$

This solution will come closer to satisfying the terminal conditions $h = 0$, provided the constants c_i are sufficiently small. The process is repeated several times until the terminal boundary conditions are satisfied to an acceptable tolerance. If, on any given iteration, $h(\bar{t}_1)$ diverges (i.e., moves away from $h = 0$), the constants c_i are scaled down until a new reference solution, better than the old one, is obtained.

A discussion of the convergence properties of this successive approximations scheme will be deferred to a later section dealing with some examples.

In the foregoing analysis, the system of Eqs. (1) was integrated to some time $\bar{t}_1 \neq t_1$ with initial conditions consistent

¶ If $p < 2n$, Eqs. (11) always possess a nontrivial solution. If $p = 2n$, $det[\partial g/\partial z]$ cannot vanish since the boundary conditions must be independent, so that $\delta z\ (t_0) = 0$. The boundary-value roblem then degenerates to an initial value problem.

with $g = 0$. The successive-approximations scheme [Eqs. (10)] involves successive approximations to the terminal time $\bar{t}_1$, as well as the initia lconditions, and requires $2n - p + 1$ integrations of the adjoint equations (4). Alternately, one might choose one of the terminal boundary conditions, say $h_{2n-p+1} = 0$, as a stopping condition to determine the terminal time. In this case, only $2n - p$ integrations of the adjoint system (4) are required. Since $h_{2n-p+1} \equiv 0$, we have

$$dh_{2n-p+1} = \left[\frac{\partial h_{2n-p+1}}{\partial z}\right]^T dz(\bar{t}_1) + \frac{\partial h_{2n-p+1}}{\partial t} d\bar{t}_1 \equiv 0$$

and, if Eqs. (4) are integrated $2n - p$ times with boundary conditions,

$$_i\lambda(\bar{t}_1) = \left[\frac{\partial h_i}{\partial z}\right] - \frac{\dot{h}_i(\bar{t}_1)}{\dot{h}_{2n-p+1}(\bar{t}_1)}\left[\frac{\partial h_{2n-p+1}}{\partial z}\right] \qquad i = 1, \ldots, 2n - p \quad (9')$$

it is easy to show that the analog of Eqs. (10) is

$$_i\lambda^T(t_0)\delta z(t_0) = dh_i \qquad i = 1, \ldots, 2n - p \quad (10')$$

This approach may be useful in some problems since it involves one less integration of the adjoint system than the scheme outlined. We are tacitly assuming, however, that the stopping condition can be attained with the guessed initial conditions.

II. Variational Boundary-Value Problem

The variational boundary-value problem will be formulated in the Mayer form.[9] It is required to determine the m vector of control variables $u(t)$, $t_0 \leq t \leq t_1$ (t_0 given), which minimizes the scalar function of terminal values

$$J[t_1, x(t_1)] \quad (12)$$

subject to the n differential constraints (equations of state)

$$\dot{x} = f(t, x, u) \quad (13)$$

the inequality constraints

$$\alpha(t, x, u) \leq 0 \quad (14)$$

$$\beta(t, x) \leq 0 \quad (15)$$

where α and β may be vectors as long as the constraints are consistent and subject to the end conditions

$$\xi[t_0, x(t_0)] = 0 \quad (16)$$

$$\eta[t_1, x(t_1)] = 0 \quad (17)$$

where ξ is an r vector, and η is a q vector ($q < 2n - r + 1$).

Before proceeding to formulate the variational boundary-value problem, we shall dispense with the inequality constraints [Eqs. (14) and (15)] in the following manner. Similarly to Ref. 6, we define

$$\phi_i = \begin{cases} \alpha_i^4(t, x, u) & \alpha_i > 0 \\ 0, & \alpha_i \leq 0 \end{cases} \quad (18)$$

$$\psi_i = \begin{cases} \beta_i^4(t, x) & \beta_i > 0 \\ 0, & \beta_i \leq 0 \end{cases} \quad (19)$$

and add the following differential constraints to the problem

$$\dot{y} = \phi \quad (20)$$

$$\dot{w} = \psi \quad (21)$$

as well as the end conditions

$$y(t_0) = y(t_1) = 0 \qquad w(t_0) = w(t_1) = 0 \quad (22)$$

We note that ϕ_i and ψ_i are three times continuously differentiable in all arguments at $\alpha_i = 0$ and $\beta_i = 0$, respectively, if α_i and β_i are.

End conditions (22) are clearly equivalent to the inequality constraints (14) and (15) since ϕ_i and ψ_i are nonnegative. In practice it is difficult, if not impossible, to obtain $y(t_1) = 0$ and $w(t_1) = 0$ exactly in problems with $\alpha = 0$ and $\beta = 0$ subarcs. This means that inequality constraints (14) and (15) will not be satisfied exactly. Small violations of these constraints will occur. It is clearly possible to formulate constraints (14) and (15) so that small violations are acceptable. We will see in Sec. III that such small violations are necessary so that this formulation of inequality constraints can be used in neighboring extremal control.

This formulation of inequality constraints has increased the dimension of the state x by the dimension of y plus the dimension of w. End conditions (22) are, however, of the form (16) and (17), so that, without loss of generality, we may now treat the Mayer problem defined by Eqs. (12, 13, 16, and 17).

The Euler-Lagrange equations that must be satisfied on the extremal are

$$\dot{\mu} = -[\partial f/\partial x]^T\mu \quad (23)$$

$$[\partial f/\partial u]^T\mu = 0 \quad (24)$$

where

$$\left[\frac{\partial f}{\partial x}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \qquad \left[\frac{\partial f}{\partial u}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}$$

and μ_i are the n Lagrange multipliers. Further, the following boundary conditions must be satisfied. At t_0 we must have

$$\mu^T(t_0)dx(t_0) = 0 \quad (25)$$

$$d\xi(t_0) = [\partial\xi/\partial x]dx(t_0) = 0 \quad (26)$$

where

$$\left[\frac{\partial \xi}{\partial x}\right] = \begin{bmatrix} \frac{\partial \xi_1}{\partial x_1} & \cdots & \frac{\partial \xi_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \xi_r}{\partial x_1} & \cdots & \frac{\partial \xi_r}{\partial x_n} \end{bmatrix}_{t_0}$$

Equations (25) and (26) have the following interpretation. Equation (26) is solved for r of the $dx_i(t_0)$, and these are eliminated from Eq. (25). The coefficients of the remaining $n - r$ $dx_i(t_0)$ are then set equal to zero in Eq. (25), thus yielding $n - r$ initial boundary conditions on the Lagrange multipliers.

At t_1 we must have

$$\left[\frac{\partial J}{\partial x}\right]^T dx(t_1) + \frac{\partial J}{\partial t}dt_1 + \mu^T(t_1)dx(t_1) - [\mu^T(t_1)f(t_1)]dt_1 = 0 \quad (27)$$

$$d\eta(t_1) = \left[\frac{\partial \eta}{\partial x}\right]dx(t_1) + \left[\frac{\partial \eta}{\partial t}\right]dt_1 = 0 \quad (28)$$

where

$$\left[\frac{\partial \eta}{\partial x}\right] = \begin{bmatrix} \frac{\partial \eta_1}{\partial x_1} & \cdots & \frac{\partial \eta_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \eta_q}{\partial x_1} & \cdots & \frac{\partial \eta_q}{\partial x_n} \end{bmatrix}_{t_1} \qquad \left[\frac{\partial \eta}{\partial t}\right] = \begin{bmatrix} \frac{\partial \eta_1}{\partial t} \\ \vdots \\ \frac{\partial \eta_q}{\partial t} \end{bmatrix}_{t_1}$$

$$\left[\frac{\partial J}{\partial x}\right] = \begin{bmatrix} \frac{\partial J}{\partial x_1} \\ \vdots \\ \frac{\partial J}{\partial x_n} \end{bmatrix}$$

Equations (27) and (28) have the following interpretation. Equation (28) is solved for q of the $dx_i(t_1)$, dt_1, and these are eliminated from Eq. (27). The coefficients of the remaining $n + 1 - q\, dx_i(t_1)$, dt_1 are then set equal to zero in Eq. (27), thus yielding $n + 1 - q$ terminal boundary condition involving the Lagrange multipliers.

The variational boundary-value problem therefore consists of the $2n$ differential equations (13) and (23), where the controls $u(t)$ are determined from Eq. (24). The boundary conditions are as follows: r initial conditions specified in Eq. (16) and $n - r$ initial conditions arising from Eqs. (25) and (26)—a total of n initial conditions; q terminal conditions specified in Eq. (17) and $n + 1 - q$ terminal conditions arising from Eqs. (27) and (28)—a total of $n + 1$ terminal conditions. We therefore have a total of $2n + 1$ boundary conditions, and the variational boundary-value problem is exactly of the form treated in Sec. I with the identifications $z^T = [x^T, \mu^T]$ and $p = n$.

We now write down the $2n$ equations adjoint to the linear perturbation equations of the variational boundary-value problem [see Eq. (4)], noting that

$$f = f[t, x, u(t, x, \mu)]$$

since the control is implicitly eliminated via the Euler-Lagrange equations in the controls (24). Let the $2n$ vector of adjoint variables λ be

$$\lambda = \begin{bmatrix} \lambda^x \\ \hline \lambda^\mu \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \\ \hline \lambda_{n+1} \\ \vdots \\ \lambda_{2n} \end{bmatrix}$$

then

$$\dot{\lambda}^x = -\left\{\left[\frac{\partial f}{\partial x}\right]^T + \left[\frac{\partial u}{\partial x}\right]^T\left[\frac{\partial f}{\partial u}\right]^T\right\}\lambda^x + \left\{\left[\frac{\partial^2 H}{\partial x^2}\right] + \left[\frac{\partial u}{\partial x}\right]^T\left[\frac{\partial^2 H}{\partial u \partial x}\right]\right\}\lambda^\mu$$

$$\dot{\lambda}^\mu = -\left[\frac{\partial u}{\partial \mu}\right]^T\left[\frac{\partial f}{\partial u}\right]^T\lambda^x + \left\{\left[\frac{\partial f}{\partial x}\right] + \left[\frac{\partial u}{\partial \mu}\right]^T \times \left[\frac{\partial^2 H}{\partial u \partial x}\right]\right\}\lambda^\mu \quad (29)$$

where $H = \mu^T f$ = Hamiltonian,

$$\left[\frac{\partial u}{\partial x}\right] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial u_m}{\partial x_1} & \cdots & \frac{\partial u_m}{\partial x_n} \end{bmatrix} \qquad \left[\frac{\partial u}{\partial \mu}\right] = \begin{bmatrix} \frac{\partial u_1}{\partial \mu_1} & \cdots & \frac{\partial u_1}{\partial \mu_n} \\ \vdots & & \vdots \\ \frac{\partial u_m}{\partial \mu_1} & \cdots & \frac{\partial u_m}{\partial \mu_n} \end{bmatrix}$$

$$\left[\frac{\partial^2 H}{\partial u \partial x}\right] = \begin{bmatrix} \frac{\partial^2 H}{\partial u_1 \partial x_1} & \cdots & \frac{\partial^2 H}{\partial u_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 H}{\partial u_m \partial x_1} & \cdots & \frac{\partial^2 H}{\partial u_m \partial x_n} \end{bmatrix} = \frac{\partial}{\partial u}\left[\frac{\partial H}{\partial x}\right]^T =$$

$$\begin{bmatrix} \frac{\partial}{\partial u_1} \\ \vdots \\ \frac{\partial}{\partial u_m} \end{bmatrix}\begin{bmatrix} \frac{\partial H}{\partial x_1} & \cdots & \frac{\partial H}{\partial x_n} \end{bmatrix} = \left[\frac{\partial^2 H}{\partial x \partial u}\right]^T$$

The two matrices of partial derivatives, $[\partial u/\partial x]$ and $[\partial u/\partial \mu]$, are obtained from differentiation of the Euler-Lagrange equation (24) in the controls

$$\left[\frac{\partial^2 H}{\partial x \partial u}\right] + \left[\frac{\partial u}{\partial x}\right]^T\left[\frac{\partial^2 H}{\partial u^2}\right] = 0 \quad (30)$$

$$\left[\frac{\partial f}{\partial u}\right] + \left[\frac{\partial u}{\partial \mu}\right]^T\left[\frac{\partial^2 H}{\partial u^2}\right] = 0 \quad (31)$$

Assuming $[\partial^2 H/\partial u^2]$ is nonsingular, which excludes extremals with singular subarcs, we obtain

$$\left[\frac{\partial u}{\partial x}\right]^T = -\left[\frac{\partial^2 H}{\partial x \partial u}\right]\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1} \quad (32)$$

$$\left[\frac{\partial u}{\partial \mu}\right]^T = -\left[\frac{\partial f}{\partial u}\right]\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1} \quad (33)$$

The final form of the adjoint equations becomes

$$\dot{\lambda} = \left[\begin{array}{c|c} -\left[\frac{\partial f}{\partial x}\right]^T + \left[\frac{\partial^2 H}{\partial x \partial u}\right]\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left[\frac{\partial f}{\partial u}\right]^T & \left[\frac{\partial^2 H}{\partial x^2}\right] - \left[\frac{\partial^2 H}{\partial x \partial u}\right]\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left[\frac{\partial^2 H}{\partial u \partial x}\right] \\ \hline \left[\frac{\partial f}{\partial u}\right]\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left[\frac{\partial f}{\partial u}\right]^T & \left[\frac{\partial f}{\partial x}\right] - \left[\frac{\partial f}{\partial u}\right]\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left[\frac{\partial^2 H}{\partial u \partial x}\right] \end{array}\right]\lambda \quad (34)$$

It is observed that the upper left-hand partition of the matrix in Eq. (34) is just the negative transpose of the lower right-hand partition. The lower left-hand and upper right-hand partitions are symmetric.

Equations (34) are integrated $n + 1$ times each iteration [i.e., each time Eqs. (10) are used to determine an improved set of initial conditions and terminal time], since $n + 1$ terminal boundary conditions are specified for the variational boundary-value problem. It is clear that for fixed time variational problems only n integrations are required. In this case there are n terminal boundary conditions, Eqs. (10) are n in numbers, and $d\bar{t}_1 = 0$. In general, only n integrations of Eqs. (34) are required if one of the terminal boundary conditions is singled out as a stopping condition [see Eqs. (9′) and (10′)].

We note that the requirement that $[\partial^2 H/\partial u^2]$ be nonsingular excludes systems linear in the control. Magnitude constraints on the control variables, which are a special case of constraints (14), can be handled within the general framework discussed previously.

III. Linear Neighboring Extremal Control

The analysis in Secs. I and II enables us to determine the open-loop control $u(t)$ for a wide class of optimization problems. Here we are interested in linear feedback control, in the neighborhood of an optimal nominal trajectory, in the presence of small disturbances. We assume that the state can be measured exactly, thus avoiding the filtering problem. We will see that once having solved the open-loop problem by the method of Secs. I and II, we have all the ingredients necessary to construct the matrix of linear, time-varying, feedback gains for neighboring extremal control.

Assume that the open-loop problem has been solved. We recall that the first n boundary-value problem variables are the state variables $x(t)$, and the second n boundary-value problem variables are the Lagrange multipliers $\mu(t)$. We also recall that n initial boundary conditions and $n + 1$

terminal boundary conditions are imposed on the variational boundary-value problem. (If final time t_1 is specified, n terminal boundary conditions are given.) We now distinguish between the end conditions imposed (17) and the terminal boundary conditions arising from the variational formulation (27) and (28) and write the $n + 1$ terminal boundary conditions

$$h = \begin{bmatrix} \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} \eta_1[t_1, x(t_1)] \\ \vdots \\ \eta_q[t_1, x(t_1)] \\ \zeta_{q+1}[t_1, x(t_1), \mu(t_1)] \\ \vdots \\ \zeta_{n+1}[t_1, x(t_1), \mu(t_1)] \end{bmatrix} \tag{35}$$

Equations (10) may therefore be written

$$\begin{aligned} \dot{\eta}_1(t_1)dt_1 + {}_1\lambda^{xT}(t)\delta x(t) + {}_1\lambda^{\mu T}(t)\delta\mu(t) &= d\eta_1 \\ &\vdots \\ \dot{\eta}_q(t_1)dt_1 + {}_q\lambda^{xT}(t)\delta x(t) + {}_q\lambda^{\mu T}(t)\delta\mu(t) &= d\eta_q \\ \dot{\zeta}_{q+1}(t_1)dt_1 + {}_{q+1}\lambda^{xT}(t)\delta x(t) + {}_{q+1}\lambda^{T\mu}(t)\delta\mu(t) &= d\zeta_{q+1} \\ &\vdots \\ \dot{\zeta}_{n+1}(t_1)dt_1 + {}_{n+1}\lambda^{Tx}(t)\delta x(t) + {}_{n+1}\lambda^{T\mu}(t)\delta\mu(t) &= d\zeta_{n+1} \end{aligned} \tag{36}$$

where the initial time t_0 has been generalized, since any time may be considered as initial time. If we set $d\eta = d\zeta = 0$ in Eq. (36), we clearly have relations between instantaneous perturbations in the state, the Lagrange multipliers, and terminal time on neighboring extremals to the same end point. (In what follows, we could allow $d\eta \neq 0$ and consider neighboring extremals to a neighboring end point, but this is a very trivial extension.)

Without loss of generality, we assume $\dot{\zeta}_{n+1}(t_1) \neq 0$ and solve for dt_1:

$$dt_1 = -\frac{1}{\dot{\zeta}_{n+1}(t_1)}[{}_{n+1}\lambda^{xT}(t)\delta x(t) + {}_{n+1}\lambda^{\mu T}(t)\delta\mu(t)] \tag{37}$$

and eliminate dt_1 from the remaining Eq. (36). Defining

$$\Lambda_1(t) = \begin{bmatrix} {}_1\lambda^{xT}(t) - \dfrac{\dot{\eta}_1(t_1)}{\dot{\zeta}_{n+1}(t_1)}\,{}_{n+1}\lambda^{xT}(t) \\ \vdots \\ {}_n\lambda^{xT}(t) - \dfrac{\dot{\zeta}_n(t_1)}{\dot{\zeta}_{n+1}(t_1)}\,{}_{n+1}\lambda^{xT}(t) \end{bmatrix}$$

$$\Lambda_2(t) = \begin{bmatrix} {}_1\lambda^{T\mu}(t) - \dfrac{\dot{\eta}_1(t_1)}{\dot{\zeta}_{n+1}(t_1)}\,{}_{n+1}\lambda^{\mu T}(t) \\ \vdots \\ {}_n\lambda^{T\mu}(t) - \dfrac{\dot{\zeta}_n(t_1)}{\dot{\zeta}_{n+1}(t_1)}\,{}_{n+1}\lambda^{\mu T}(t) \end{bmatrix}^{**}$$

we obtain

$$\Lambda_1(t)\,\delta x(t) + \Lambda_2(t)\,\delta\mu(t) = 0 \tag{38}$$

and assuming Λ_2 to be nonsingular,††

$$\delta\mu(t) = -\Lambda_2^{-1}(t)\Lambda_1(t)\delta x(t) \tag{39}$$

** The form of the rows of matrices Λ_1 and Λ_2 motivates the choice of boundary conditions on the adjoint variables in Eq. (9′).

†† This inversion will be discussed in a later section dealing with numerical results.

On a neighboring extremal, the perturbed Euler-Lagrange equations in the controls (24) hold

$$\left[\frac{\partial f}{\partial u}\right]^T \delta\mu + \left[\frac{\partial^2 H}{\partial u \partial x}\right]\delta x + \left[\frac{\partial^2 H}{\partial u^2}\right]\delta u = 0 \tag{40}$$

Eliminating $\delta\mu$ with the aid of Eq. (39), we obtain

$$\delta u(t) = G(t)\,\delta x(t) \tag{41}$$

where

$$G(t) = -\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left\{\left[\frac{\partial^2 H}{\partial u \partial x}\right] - \left[\frac{\partial f}{\partial u}\right]^T \Lambda_2^{-1}\Lambda_1\right\} \tag{42}$$

is the matrix of linear, time-varying, feedback gains for neighboring extremal control. Here we have again assumed the nonvanishing of $det\,[\partial^2 H/\partial u^2]$.

If neighboring extremal control to a neighboring end point is desired [$d\eta \neq 0$ in Eqs. (36)], it is very easy to show that the control law (41) becomes

$$\delta u(t) = G(t)\delta x(t) + N(t)d\eta' \tag{41'}$$

where

$$d\eta'^T = [d\eta_1, \ldots, d\eta_q, \overbrace{0, \ldots, 0}^{n-q}]$$

$$N(t) = -\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left[\frac{\partial f}{\partial u}\right]^T \Lambda_2^{-1}$$

Here $d\eta^T = [d\eta_1, \ldots, d\eta_q]$ defines the neighboring end point and may be specified (or changed) anywhere along the system (13) trajectory as the control requirements change.

As was asserted earlier, all the components necessary to construct $G(t)$ and $N(t)$ are available from the boundary-value problem solution. It is noted that $G(t)$ and $N(t)$ are evaluated, at frequent enough intervals of time to describe them as continuous functions, along an optimal nominal trajectory. This is done prior to the control scheme implementation, i.e., not in real time. The method of implementation of this scheme has been amply discussed in Ref. 8 and will not be repeated here.

This scheme [Eqs. (35–42)] of constructing the gains requires $n + 1$ integrations of the adjoint equations (34), solutions that are available from the boundary-value problem solution. If one of the terminal boundary conditions is used as a stopping condition, then Eq. (10′) replaces Eq. (36), and only n solutions of the adjoint equations are required. The number of required adjoint solutions is the same for neighboring extremal control to the same end point (41) and to a neighboring end point (41′).

In the scheme proposed by Breakwell, Speyer, and Bryson,[8] a system adjoint to the adjoint equations (34), i.e., the perturbation equations, is integrated $n + q$ times to obtain the gains for neighboring extremal control to a neighboring end point. Here q is the number of end conditions (17) defining the end point. The present scheme therefore requires $q - 1$ [q if Eqs. (10′) are used] fewer integrations of an equivalent system of equations. This represents a considerable computation time saving, as Eqs. (34) are quite involved even for the simplest problems.

A further simplification results in minimum time problems. For minimum time problems, it can be shown that

$$dt_1 = -\mu^T(t)\delta x(t) \tag{43}$$

to the first order. It is convenient to use this relation to eliminate dt_1 from Eqs. (36) when the Lagrange multipliers can be scaled to trade a terminal boundary condition for an initial boundary condition (for the variational boundary-value problem), as is done in the example of Sec. V. In this

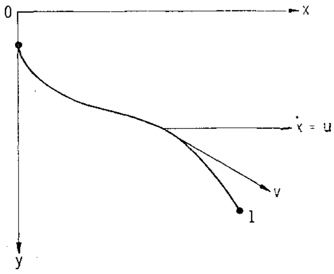

Fig. 1 **Brachistochrone problem.**

case, n integrations of the adjoint equations are required, and the gains matrix becomes

$$G(t) = -\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left\{\left[\frac{\partial^2 H}{\partial u \partial x}\right] - \left[\frac{\partial f}{\partial u}\right]^T \Lambda_4^{-1}\Lambda_3\right\} \quad (44)$$

where

$$\Lambda_3(t) = \begin{bmatrix} \lambda^{xT}(t) - \dot{\eta}_1(t_1)\mu^T(t) \\ \cdot \\ \cdot \\ \cdot \\ {}_n\lambda^{xT}(t) - \dot{\zeta}_n(t_1)\mu^T(t) \end{bmatrix} \qquad \Lambda_4(t) = \begin{bmatrix} {}_1\lambda^{\mu T}(t) \\ \cdot \\ \cdot \\ \cdot \\ {}_n\lambda^{\mu T}(t) \end{bmatrix}$$

The last boundary condition in Eq. (35) has been eliminated, and therefore the last of Eqs. (36) is absent in this case.

An evident shortcoming of the control scheme outlined previously is that there are no gains for times greater than the nominal terminal time, i.e., for $dt_1 > 0$. Also, the gains often become unbounded at the nominal terminal time, as will be seen in the example of Sec. VI. When this occurs, the linearity assumptions on which the foregoing analysis is based are clearly violated. These shortcomings may be quite serious if dt_1 is positive and large on a particular trajectory. These terminal aspects of the present control scheme require further investigation.

Before proceeding to numerical examples of the optimization and control schemes, we point out the implications of the formulation of inequality constraints on the control implementation. If, for example, one constraint of the form (14) is imposed, then the dimension of the state is increased by one, specifically,

$$x^T = [x_1, \ldots, x_n, y_1]$$

$$\delta x^T = [\delta x_1, \ldots, \delta x_n, y_1]$$

which is required in the control law. If more constraints are imposed, their associated differential equations (20) and (21) must also be integrated. These equations are generally very simple, and their integration does not appear to pose a serious problem. Because of the formulation of inequality constraints (18–22), the trajectory is insensitive to the constraints unless a violation actually occurs. Small violations of the constraints must therefore be present on the nominal trajectory so that it will be sensitive to violations on a neighboring trajectory.

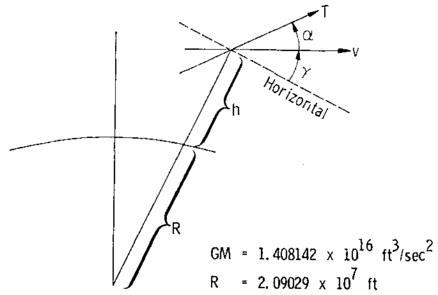

Fig. 3 Planar rocket motion.

Handling inequality constraints in this way eliminates the problem of determining which constraint is operative and when to leave one and follow another, when more than one constraint is present.

IV. A Brachistochrone Problem

It is interesting to observe the inability of a steepest-descent optimization scheme to generate a Eulerian control even for a simple brachistochrone problem. We seek the shape of a "frictionless tube" joining point 0 to point 1 (see Fig. 1) such that a particle falling through this tube in a constant gravitational field will do so in a shorter time than through any other such tube joining these two points. This tube will be "shaped" with the aid of a horizontal velocity component $u(t)$.

Energy must be conserved, so that

$$v^2 = 2gy$$

where g is the constant gravitational acceleration, taken as 5 in this problem. But

$$v^2 = \dot{x}^2 + \dot{y}^2$$

so that the equations of state are

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= (2gy - u^2)^{1/2} \end{aligned} \quad (45)$$

The particular problem considered here is to find that control

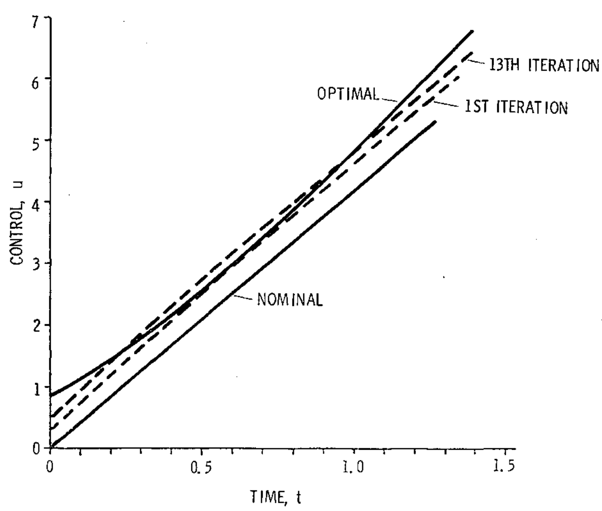

Fig. 2 Brachistochrone—control by steepest descent.

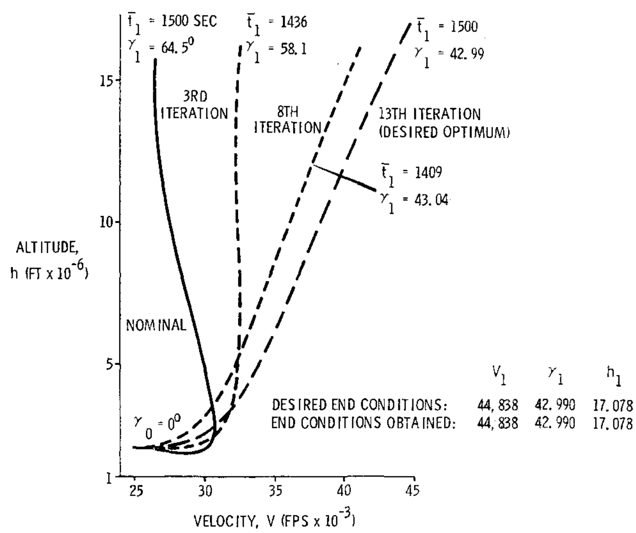

Fig. 4 Minimum time to escape V, γ, h (h vs V plot).

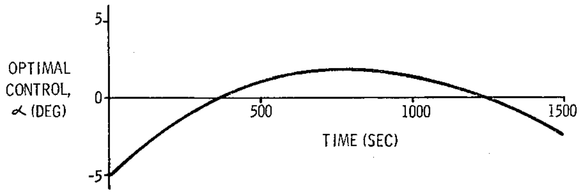

Fig. 5 Minimum time to escape V, γ, h—control α.

$u(t)$ that will result in a least time of fall from $(x_0, y_0) = (0, 1)$ to $(x_1, y_1) = (5, 8)$.

The Eulerian solution was obtained via the method described in this paper and is marked "optimal" in Fig. 2. Boundary conditions were satisfied to six significant figures. A steepest descent was started with the nominal control marked in Fig. 2. End conditions were satisfied rapidly, but the control obtained from the thirteenth iteration does not match the Eulerian control well. Further steepest-descent iterations do not differ significantly from the thirteenth. It is clear that the nominal control for neighboring extremal control should not be obtained by steepest descent.

V. Optimal Planar Out-of-Atmosphere Rocket Trajectories

The system model to be considered here and in the next section is the planar motion of a rocket vehicle in an inverse square field. The equations of motion may be written

$$\left.\begin{aligned} \dot{v} &= \frac{T\cos\alpha}{m} - \frac{GM}{(R+h)^2}\sin\gamma \\ \dot{\gamma} &= \frac{T\sin\alpha}{mv} + \frac{v\cos\gamma}{(R+h)} - \frac{\mathrm{GM}\cos\gamma}{(R+h)^2 v} \\ \dot{h} &= v\sin\gamma \end{aligned}\right\} \tag{46}$$

where v is the total velocity, γ the flight path angle, h the altitude, T the thrust, m the mass, GM the gravitational constant times the planet mass, R the planet radius, and α the angle between the velocity and thrust directions (see Fig. 3). The state vector $x^T = (v, \gamma, h)$; the control $u = \alpha$. Thrust and mass are specified functions of time.

The Euler-Lagrange (23) and (24) and adjoint equations (34) will not be written down because of their length. They follow directly from the general forms presented in Sec. II.

We will be concerned here with minimum time‡‡ problems from a given initial to a given terminal state. Accordingly, the end conditions (16) and (17) are

$$\begin{aligned} x_1(t_0) &= v(t_0) = v_0 & x_1(t_1) &= v_1 \\ x_2(t_0) &= \gamma(t_0) = \gamma_0 & x_2(t_1) &= \gamma_1 \\ x_3(t_0) &= h(t_0) = h_0 & x_3(t_1) &= h_1 \end{aligned} \tag{47}$$

Equations (25) and (26) yield no initial boundary conditions. Equations (27) and (28) yield the terminal boundary condition

$$\mu_1(t_1)\dot{v}(t_1) + \mu_2(t_1)\dot{\gamma}(t_1) + \mu_3(t_1)\dot{h}(t_1) = 1 \tag{48}$$

where μ_1, μ_2, and μ_3 are the Lagrange multipliers associated with the velocity, path angle, and altitude differential constraints (46), respectively. We take advantage of the linearity and homogeneity of the Euler-Lagrange equations in the Lagrange multipliers and scale those equations by $s/\mu_1(t_0)$, where s is an arbitrary positive constant. Denoting the scaled multipliers by a bar, we see that

$$\bar{\mu}_1(t_0) = [s/\mu_1(t_0)] \times \mu_1(t_0) = s \tag{49}$$

We take condition (49) instead of (48) as the seventh boundary condition. Since $t_0 = 0$ is specified, the seven required boundary conditions are (47) and (49). We can recover the actual values of the Lagrange multipliers via (48), since

$$\bar{\mu}_1(t_1)\dot{v}(t_1) + \bar{\mu}_2(t_1)\dot{\gamma}(t_1) + \bar{\mu}_3(t_1)\dot{h}(t_1) = s/\mu_1(t_0)$$

To demonstrate the convergence of the present optimization scheme, consider the problem of steering a 100-lb vehicle with a constant thrust of 30 lb and a fuel flow rate of 0.05 lb/sec ($I_{sp} = 600$) from a circular orbit around the earth at an altitude of $\frac{1}{10}$ earth radius to the escape end conditions given in Fig. 4. An initial guess of final time and the two Lagrange multipliers $\bar{\mu}_2(t_0)$ and $\bar{\mu}_3(t_0)$ resulted in the trajectory marked nominal in Fig. 4. Subsequent improved trajectories are shown. The desired end conditions were satisfied to six significant figures on the thirteenth iteration. The optimal control (thirteenth iteration) is given in Fig. 5. In the iteration procedure, the desired changes in end conditions $dh_i(\bar{t}_1)$ were initially set at $-0.2\ h_i(\bar{t}_1)$. The constant 0.2 was systematically increased to 1.0 in subsequent iterations.

The convergence shown for this example is typical of out-of-atmosphere rocket trajectories in two and three dimensions. A good initial guess of the initial values of the Lagrange multipliers is not required. The present scheme is extremely effective for this type of problem and is superior to steepest-descent procedures in terms of rates of convergence and total computing time. This scheme has yet to be fully tested for dissipative systems, such as atmospheric rocket trajectories, where serious instabilities may arise. In such

‡‡ For nonvanishing thrust, this is equivalent to minimum fuel expenditure or maximum payload.

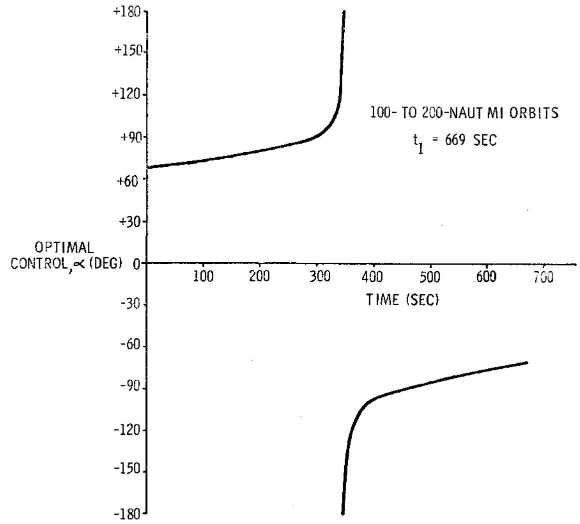

Fig. 6 Circular orbits transfer.

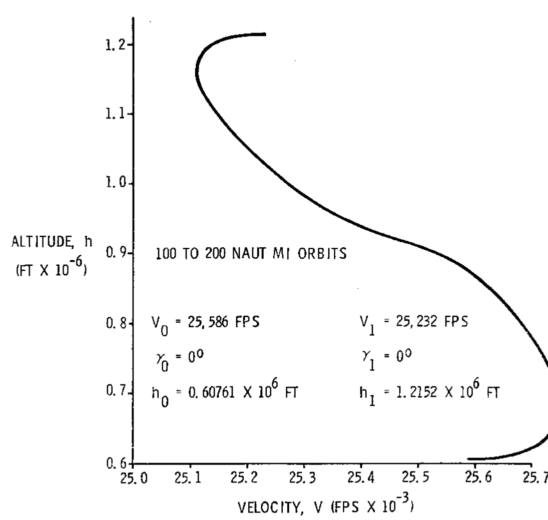

Fig. 7 Optimal circular orbits transfer—h vs V plot.

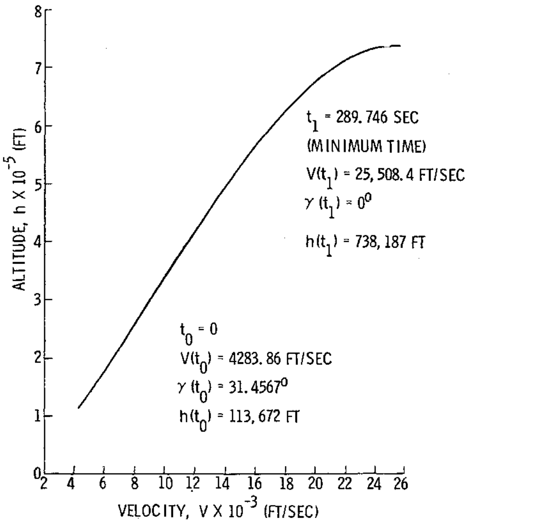

Fig. 8 Optimal nominal circular orbit injection (h vs V plot).

problems, extremely accurate integration of the adjoint equations is required.

As a further example, a minimum-time circular orbit transfer was computed for a 200-lb vehicle with the same thrust characteristics as those of the previous problem. The optimal control is given in Fig. 6. The optimal control is continuous, contrary to a previous report in the literature.[3,10] An altitude-velocity plot of the optimal trajectory is shown in Fig. 7. It may be mentioned that all the optimal controls shown in this paper satisfy the Legendre-Clebsch necessary condition.

VI. Neighboring Extremal Control of Circular Orbit Injection

Here we consider a circular orbit injection of a Nova class second stage from first-stage burnout conditions. A constant acceleration (T/M = 80.57 ft/sec²) vehicle is considered. The burnout conditions (at $t_0 = 0$) and circular orbit end conditions are given in Fig. 8, which shows the optimal (minimum-time) trajectory, which is to be used as the nominal. [Planar motion is assumed; Eqs. (46) are the equations of motion.]

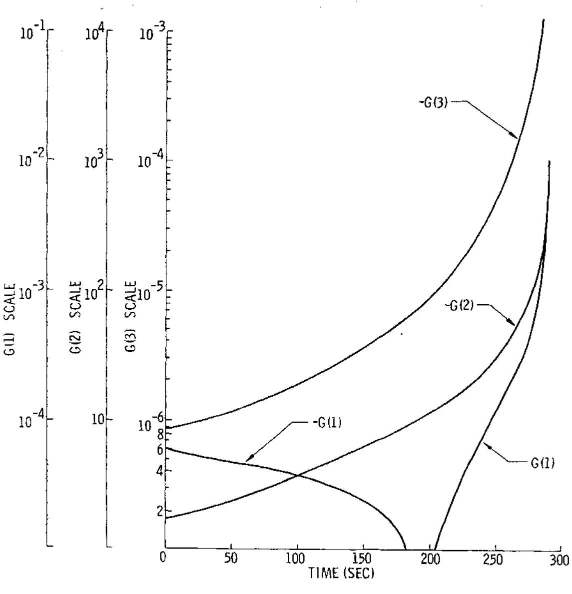

Fig. 9 Elements of gains matrix.

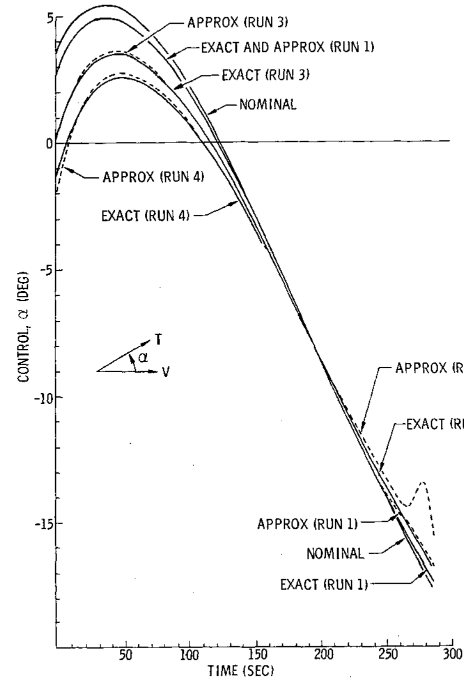

Fig. 10 Exact and approximate optimal control.

Since this problem involves one control variable, the gains matrix is 1 × 3. The gains matrix of the form (44) is used. The control law is

$$\delta\alpha(t) = [G_1(t), G_2(t), G_3(t)] \begin{bmatrix} \delta v(t) \\ \delta\gamma(t) \\ \delta h(t) \end{bmatrix} \tag{50}$$

The three gains are shown in Fig. 9.

It is noted that $\Lambda_4(t_l) = 0$ in view of the boundary conditions specified on the adjoint variables (9) which, for this problem, are

$$_1\lambda(t_l) = \begin{bmatrix} {}_1\lambda^x(t_l) \\ \\ --- \\ \\ {}_1\lambda^\mu(t_l) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ --- \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad {}_2\lambda(t_l) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad {}_3\lambda(t_l) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{51}$$

This results in infinite gains at the nominal terminal time and is due to the hard terminal constraints on velocity, path angle, and altitude. A plausible explanation for this behavior is that large controls are required to null even small errors in a very short time. In practice, control bounds are, of course, imposed, and, when near the terminal time, these bounds are exceeded because of the growing gains; control is simply held on the bound. Although a feasible solution to circumvent the difficulties, this procedure may result in poor terminal control accuracy.

This control scheme was simulated on a digital computer for several combinations of initial errors in velocity, path angle, and altitude, given in Table 1. Control was terminated at terminal times given by Eq. (43), which is a linear estimate of the minimum time on the neighboring extremals. The resultant terminal errors, terminal times, and the actual

Table 1 Control runs

Run	Initial errors[a] δV_0, fps	$\delta\gamma_0$, deg	δh_0, ft	Terminal errors dV_1, fps	$d\gamma_1$, deg	dh_1, ft	Terminal time, sec	Minimum time, sec
1	10.0	0.5	100.0	0.3	0.005	−1.0	289.565	289.568
2	30.0	1.0	300.0	3.1	0.056	8.0	289.236	289.253
3	50.0	2.0	500.0	0.6	0.054	35.0	288.874	288.910
4	75.0	3.0	700.0	−3.9	0.079	53.0	288.441	288.588

[a]Error ≡ actual − optimal nominal.

minimum times on the neighboring extremals for each given set of initial errors are also given in Table 1.

The optimization scheme (discussed in Secs. I and II) is extremely efficient in generating neighboring extremals for slightly different initial (or terminal) boundary conditions. Extremals were thus generated, with at most two iterations each, for each set of initial conditions defined by the initial errors in Table 1. The neighboring controls, as well as the nominal control, are marked "exact" in Fig. 10. Also plotted are the control programs using the control law (50). These are marked "approximate."

It is seen that, with only one nominal trajectory, the neighboring extremal control scheme is able to successfully handle quite a wide range of initial errors. It is noted, however, that control simulations with $dt_1 > 0$ were purposely avoided. A more systematic series of simulation, as well as a definition of control system requirements, is necessary to define the range of initial errors this control scheme can successfully handle for a given control problem. It is of course possible to use several nominal trajectories and their associated control gains if a very wide range of initial errors in state is expected.

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